Please check the examination details below before entering your candidate information

| Candidate surname | Other names |
|---|---|
| | |

**Pearson Edexcel Level 3 GCE**

Centre Number | Candidate Number

# Thursday 6 June 2019

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **9FM0/02**

# Further Mathematics

## Advanced

## Paper 2: Core Pure Mathematics 2

**You must have:**
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

**Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

## Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

## Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

## Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

*Turn over* ▶

**Answer ALL questions. Write your answers in the spaces provided.**

**1.** (a) Prove that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -k < x < k$$

stating the value of the constant $k$.

**(5)**

(b) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln\sqrt{2-3x}\right)$$

**(5)**

(a) notice we are asked to prove the definition of an inverse hyperbolic function (one of the results given in the formula booklet)

WAY 1: manipulating LHS to give RHS

first let $y = \tanh^{-1}(x)$

taking tanh of both sides

$\tanh y = x \Rightarrow x = \tanh y$

rewriting RHS of above using exponential definition of sinhy and coshy (or straight away memorised $\tanh y = \frac{e^{2y}-1}{e^{2y}+1}$)

$$x = \frac{\sinh y}{\cosh y} = \frac{\cancel{\tfrac{1}{2}}(e^{y}-e^{-y})}{\cancel{\tfrac{1}{2}}(e^{y}+e^{-y})} = \frac{e^{y}-e^{-y}}{e^{y}+e^{-y}} \begin{matrix}\times e^{y} \\ \times e^{y}\end{matrix}$$

$$\Rightarrow x = \frac{e^{2y}-1}{e^{2y}+1}$$

cross multiply

$$x(e^{2y}+1) = e^{2y}-1$$

expand LHS

$$e^{2y}x + x = e^{2y}-1$$

remembering need equation to solve for 'y' as $\boxed{y = \tanh^{-1}(x)}$ ∴ make 'y' the subject of above;

collect like terms

$$e^{2y} - xe^{2y} = 1 + x$$

factorise $e^{2y}$ out

$$e^{2y}(1-x) = 1+x$$

$\div(1-x) \qquad \div(1-x)$

$$e^{2y} = \frac{1+x}{1-x}$$

take logs of both sides

$$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$$

$\div 2 \qquad \div 2$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

now subbing in $y = \tanh^{-1}(x)$

$$\boxed{\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$$

where can't take logs of -ves, so need restriction on the $x$:

$$\boxed{-1 < x < 1}$$

WAY 2: manipulating RHS to give LHS

given $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

taking tanh of both sides

$$\Rightarrow x = \tanh\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right)$$

now trying to manipulate the RHS to get LHS $= x$

using tanh y exponential definition on RHS - $\tanh y = \dfrac{e^{2y}-1}{e^{2y}+1}$

$$\text{RHS} = \frac{e^{2\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right)}-1}{e^{2\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right)}+1} = \frac{e^{\ln\left(\frac{1+x}{1-x}\right)}-1}{e^{\ln\left(\frac{1+x}{1-x}\right)}+1}$$

notice ln and exponential powers CANCEL to give:

$$\frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}$$ getting common denominator $$\frac{\frac{1+x-(1-x)}{1-x}}{\frac{1+x+(1-x)}{1-x}}$$

$$= \frac{\frac{2x}{1-x}}{\frac{2}{1-x}} = \frac{2x}{2} = x = \text{LHS}$$

$$\boxed{\text{LHS} = \text{RHS} \therefore \text{original conjecture must be true}}$$

↳ also can't take logs of -ves

$$\boxed{\therefore -1 < x < 1}$$

(b) Solving the equation means solving for '$x$' - notice there are x's on BOTH SIDES of the equation

Question 1 continued

...two main ways to solve :

WAY 1: taking artanh of both sides

$\tanh^{-1}(2x) = \ln\sqrt{2-3x}$

using definition of $\tanh^{-1}(x)$ proved in part (a) ($x \to 2x$) and log power rule on RHS

$\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right) = \ln(2-3x)^{1/2}$

$\Rightarrow \frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right) = \frac{1}{2}\ln(2-3x)$

equating insides of both logs

$\frac{1+2x}{1-2x} = 2-3x$

cross multiply

$1+2x = (1-2x)(2-3x)$

$\Rightarrow 1+2x = 2-3x-4x+6x^2$

$\Rightarrow \boxed{6x^2-9x+1=0}$

WAY 2: using hyperbolic tanhx definition

$\tanh y = \frac{e^{2y}-1}{e^{2y}+1}$

$\Rightarrow 2x = \frac{e^{2\ln\sqrt{2-3x}}-1}{e^{2\ln\sqrt{2-3x}}+1}$

using log power rule

$2x = \frac{e^{\ln(\sqrt{2-3x})^2}-1}{e^{\ln(\sqrt{2-3x})^2}+1}$

notice lns and exponential powers cancel

$2x = \frac{2-3x-1}{2-3x+1}$

$\Rightarrow 2x = \frac{1-3x}{3-3x}$

cross multiply

$2x(3-3x) = 1-3x$

expand brackets

$6x-6x^2 = 1-3x$

$\boxed{\Rightarrow 6x^2-9x+1=0}$

Solving above quadratic equation for $x$

↳ calc equtn solver / quadratic formula

$x = \frac{9 \pm \sqrt{(-9)^2-4(6)(1)}}{2(6)} = \frac{9 \pm \sqrt{81-24}}{12}$

$= \frac{9 \pm \sqrt{57}}{12}$

but know $-1 < 2x < 1$

$\Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

$\Rightarrow \boxed{x = \frac{9-\sqrt{57}}{12}}$

(Total for Question 1 is 10 marks)

**2.** The roots of the equation

$$x^3 - 2x^2 + 4x - 5 = 0$$

are $p$, $q$ and $r$.

Without solving the equation, find the value of

(i) $\dfrac{2}{p} + \dfrac{2}{q} + \dfrac{2}{r}$

(ii) $(p - 4)(q - 4)(r - 4)$

(iii) $p^3 + q^3 + r^3$

**(8)**

given the cubic equation ... look to apply our roots of polynomial equations formulae:

$$\underset{\sum\alpha}{x^3 - 2x^2} + \underset{\sum\alpha\beta}{4x} - \underset{\alpha\beta\gamma}{5} = 0$$

where:

sum of roots $= \sum\alpha = -b/a$

$= -(-2)/1$

$= 2$

sum of product pairs $= \sum\alpha\beta = c/a$

$= 4/1$

$= 4$

product of roots $= \alpha\beta\gamma = -d/a$

$= -(-5)/1$

$= 5$

now we look to use the above to find related expressions asked for in the question:

(i) $\dfrac{2}{p} + \dfrac{2}{q} + \dfrac{2}{r}$ ...finding common denominator: $\dfrac{2(qr)+2(pr)+2(pq)}{pqr}$

$= \dfrac{2(\sum\alpha\beta)}{\alpha\beta\gamma} = \dfrac{2(4)}{5}$

$\boxed{= 8/5}$

Question 2 continued

(ii) $(p-4)(q-4)(r-4)$

WAY 1: have to expand these triple brackets out:

↳ first the first two brackets

$(pq-4p-4q+16)(r-4)$

expand brackets

$pqr-4pr-4qr+16r-4pq+16p+16q-64$

collect like terms

$pqr-4(pr+qr+pq)+16(p+q+r)-64$ - this can be rewritten using roots of polynomials formulae

$\alpha\beta\gamma-4(\Sigma\alpha\beta)+16(\Sigma\alpha)-64$

$5-4(4)+16(2)-64$

$=5-16+32-64$

$=\boxed{-43}$

WAY 2: treat as a 'transformation of linear roots'

i.e let $w=x-4$

$\Rightarrow x=w+4$

sub this into cubic equation

$(w+4)^3-2(w+4)^2+4(w+4)-5=0$

expand Binomially

... Pascal's triangle:

1
1 1
1 2 1
1 3 3 1

$(1w^3+3(w^2)(4)+3(w)(4)^2+1(4)^3)-2(w^2+8w+16)$

$+4w+16-5=0$

expand out

$w^3+12w^2+48w+64-2w^2-16w-32+4w+16-5=0$

collect like terms

$w^3+\underbrace{10}_{\Sigma\alpha}w^2+\underbrace{36}_{\Sigma\alpha\beta}w+\underbrace{43}_{\alpha\beta\gamma}=0$

with roots $p-4,\ q-4,\ r-4$

question asks for the product of these so using $\alpha\beta\gamma$ of transformed cubic

$=-d/a=-43/1=\boxed{-43}$

Question 2 continued

(iii) $p^3+q^3+r^3$ – this is a sum of cubes – 2 main ways to solve:

WAY 1: using memorised (Pearson textbook):

The rules for sums of cubes:

- Quadratic: $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
- Cubic: $\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)+3\alpha\beta\gamma$

$$p^3+q^3+r^3=(2)^3-3(2)(4)+3(5)$$
$$=8-24+15$$
$$=\boxed{-1}$$

WAY 2: recognising $p^3+q^3+r^3$ as a potential result of the manipulation of:

$(p+q+r)^3=(p+q+r)(p+q+r)(p+q+r)$

expand first two brackets

$=(p^2+pq+pr+pq+q^2+qr+pr+qr+r^2)(p+q+r)$

collect like terms

$=(p^2+q^2+r^2+2pq+2pr+2qr)(p+q+r)$

expand brackets

$=p^3+q^2p+r^2p+2p^2q+2p^2r+2pqr+p^2q+q^3+qr^2+2pq^2+2prq+2q^2r+$
$p^2r+q^2r+r^3+2pqr+2pr^2+2qr^2$

$\Rightarrow (p+q+r)^3=p^3+q^3+r^3+3p^2q+3p^2r+3pr^2+3qr^2+3pq^2+3q^2r+6pqr$

$\Rightarrow p^3+q^3+r^3=(p+q+r)^3-3p^2q-3p^2r-3pr^2-3qr^2-3pq^2-3q^2r-6pqr$

$$p^2+q^2+r^2=(p+q+r)^3-3(p^2q+p^2r+pr^2+qr^2+pq^2+q^2r)-6pqr$$

We want an expression with $(p+q+r)$ and need to work out what the other bracket would be. We can see by inspection that lots of terms have a squared power multiplied by an individual power, so let's start with $pq, qr, rp$ (since it's symmetrical) and we know we'll at least get some of the terms we're looking for.

$$p^3+q^3+r^3=(p+q+r)^3-3(p+q+r)(pq+pr+qr)-6pqr$$

We need to manipulate this a bit as we don't yet know whether it's correct. We expand out:

$-3(p+q+r)(pq+qr+pr)$ to get

$-3p^2q-3p^2r-3pq^2-3q^2r-3qr^2-3pr^2-9pqr$ so we've created an extra $-9pqr$ term that was not there originally which we need to counteract with a $+9pqr$

$$p^3+q^3+r^3=(p+q+r)^3-3(p+q+r)(pq+pr+qr)+9pqr-6pqr$$







P 6 1 1 7 8 A 0 6 2 8

Question 2 continued

we can collect like terms

$p^3+q^3+r^3 = (p+q+r)^3 - 3(p+q+r)(pq+qr+pr) + 3pqr$

and subbing in appropriate roots of polynomials formulae

$= (\Sigma\alpha)^3 - 3(\Sigma\alpha)(\Sigma\alpha\beta) + 3(\alpha\beta\gamma)$

$= (2)^3 - 3(2)(4) + 3(5)$

$\boxed{= -1}$

(Total for Question 2 is 8 marks)

**3.**

$$f(x) = \frac{1}{\sqrt{4x^2 + 9}}$$

(a) Using a substitution, that should be stated clearly, show that

$$\int f(x)dx = A\sinh^{-1}(Bx) + c$$

where $c$ is an arbitrary constant and $A$ and $B$ are constants to be found.

**(4)**

(b) Hence find, in exact form in terms of natural logarithms, the mean value of $f(x)$ over the interval $[0, 3]$.

**(2)**

(a) looking at fact that $\sinh^{-1}(x)$ in the answer ∴ looking at formula booklet to see which integration would involve an $\text{arsinh}(x)$

| | | |
|---|---|---|
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\arcsin\left(\frac{x}{a}\right)$ | $(\lvert x\rvert < a)$ |
| $\frac{1}{a^2+x^2}$ | $\frac{1}{a}\arctan\left(\frac{x}{a}\right)$ | |
| $\frac{1}{\sqrt{x^2-a^2}}$ | $\text{arcosh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2-a^2}\}$ | $(x > a)$ |
| $\frac{1}{\sqrt{a^2+x^2}}$ | $\text{arsinh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2+a^2}\}$ | |
| $\frac{1}{a^2-x^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right)$ | $(\lvert x\rvert < a)$ |
| $\frac{1}{x^2-a^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$ | |

but need to manipulate to just get a single 'x' - taking out $\frac{1}{\sqrt{4}} = \frac{1}{2}$

$$\frac{1}{2}\int \frac{1}{\sqrt{x^2 + 9/4}}\,dx \qquad a^2 = 9/4 \quad a = 3/2$$

2 different substitutions we can now make :

WAY 1: using substitution : $x = 3/2 \sinh u$

$$\frac{dx}{du} = 3/2\cosh u$$

$$\Rightarrow dx = 3/2\cosh u\,du$$

subbing into the integral:

$$\frac{1}{2}\int \frac{1}{\sqrt{(3/2\sinh u)^2 + 9/4}}\,3/2\cosh u\,du$$

$$= \frac{1}{2}\int \frac{3/2\cosh u}{\sqrt{\frac{9}{4}\sinh^2 u + \frac{9}{4}}}\,du$$

recognising denominator can be manipulated using the main hyperbolic identity REARRANGED: $\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow \cosh^2 x = 1 + \sinh^2 x$

$\Rightarrow \frac{9}{4}(1+\sinh^2 u) = \frac{9}{4}\cosh^2 u$

$$= \frac{1}{2}\int \frac{3/2 \cosh u}{\sqrt{9/4 \cosh^2 u}}\, du$$

$$= \frac{1}{2}\int \frac{3/2 \cosh u}{3/2 \cosh u}\, du$$

$$= \frac{1}{2}\int 1\, du$$

$$= \frac{1}{2}u + c$$

using $x = 3/2 \sinh u$

but making 'u' the subject

$\div 3/2 \qquad \div 3/2$

$\Rightarrow \frac{2}{3}x = \sinh u$

$\Rightarrow u = \operatorname{arsinh}\left(\frac{2x}{3}\right)$

subbing into answer

$$= \frac{1}{2}\left(\operatorname{arsinh}\left(\frac{2x}{3}\right)\right) + c$$

$$\boxed{= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right) + c}$$

$\Rightarrow A = 1/2,\ B = 2/3$

WAY 2: using substitution: let $x = \frac{1}{2}u$

$\frac{dx}{du} = 1/2$

$\Rightarrow dx = 1/2\, du$

subbing into integral

$$\int \frac{1}{\sqrt{4x^2+9}}\, dx = \int \frac{1}{\sqrt{4\left(\frac{1}{2}u\right)^2+9}} \times 1/2\, du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u^2+9}}\, du$$

$a^2 = 9$

square root

$\Rightarrow a = 3$

and can straight away sub into formula book integration

$$= \frac{1}{2}\operatorname{arsinh}\left(\frac{u}{3}\right) + c$$

need to sub 'u' back in ∴ rearrange $x = 1/2\, u$

$\Rightarrow u = 2x$

Question 3 continued

$$\boxed{\therefore \int f(x)\,dx = \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right) + c}$$

$\Rightarrow A = 1/2, \; B = 2/3$

(b) remembering formula to find mean value of a function

$$\overline{f(x)} = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} f(x)\,dx$$

$$= \frac{1}{3-0}\int_0^3 \frac{1}{\sqrt{4x^2+9}}\,dx$$

already know the indefinite integral from (a) - all we need to do is evaluate this at the LIMITS

$$\frac{1}{3}\left[\frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{3}\right)\right]_0^3$$

$$= \frac{1}{3}\left\{\left(\frac{1}{2}\operatorname{arsinh}(2)\right) - \frac{1}{2}\operatorname{arsinh}(0)\right)$$

$$= \frac{1}{6}\operatorname{arsinh}(2) - 0 = \frac{1}{6}\operatorname{arsinh}(2)$$

'in terms of natural logs suggests we need to use the formula book definition of

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$= \frac{1}{6}\ln\left(2 + \sqrt{2^2+1}\right)$$

$$\boxed{= \frac{1}{6}\ln\left(2 + \sqrt{5}\right)}$$

**(Total for Question 3 is 6 marks)**

**4.** The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

**(4)**

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

**(4)**

(a) WAY 1: working with exponentials

here we are dealing with complex sums of series ∴ let's use the given 'C' and 'S' expressions to get $C+iS$

$$C+iS = \cos\theta + i\sin\theta + \frac{1}{2}(\cos 5\theta + i\sin 5\theta) + \frac{1}{4}(\cos 9\theta + i\sin 9\theta) + \ldots$$

changing above into exponential form:

$$= e^{i\theta} + \frac{1}{2}e^{i5\theta} + \frac{1}{4}e^{i9\theta} + \ldots$$

now using the sum of series to infinity equation from Pure Yr 2:

$a = e^{i\theta}$

$r = \frac{1}{2}e^{i4\theta}$ convergent $\Rightarrow |r| < 1$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

$$= \frac{e^{i\theta}}{1 - \frac{1}{2}e^{i4\theta}} \quad \begin{matrix}\times 2 \\ \times 2\end{matrix}$$

$$\boxed{= \frac{2e^{i\theta}}{2 - e^{i4\theta}}}$$

WAY 2: working with mod-arg form:

from WAY 1:

$C+iS = \cos\theta + i\sin\theta + \frac{1}{2}(\cos 5\theta + i\sin 5\theta) + \frac{1}{4}(\cos 9\theta + i\sin 9\theta) + \ldots$

which using DMT can be rewritten as:

$= \cos\theta + i\sin\theta + \frac{1}{2}(\cos\theta + i\sin\theta)^5 + \frac{1}{4}(\cos\theta + i\sin\theta)^9 + \ldots$

and subbing into Pure Yr 2 sum of series to infinity

$a = \cos\theta + i\sin\theta$

$r = \frac{1}{2}(\cos\theta + i\sin\theta)^4$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

$$= \frac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)^4}$$

now changing to exponential form

$$\frac{e^{i\theta}}{1 - \frac{1}{2}e^{i4\theta}} \begin{matrix}\times 2\\ \times 2\end{matrix} = \boxed{\frac{2e^{i\theta}}{2 - e^{i4\theta}}}$$

(b) WAY 1: working with exponentials

manipulating part (a) using hyperbolic expression methods

notice we have a $2-$ in front of the exponential in the denominator; this is in the form $k+/k-/-k$ ∴ have to multiply numerator and denominator by the denominator with the exponential power NEGATED ∴ $2-e^{i-4\theta}$

$$\Rightarrow \frac{2e^{i\theta}}{2-e^{i4\theta}} \begin{matrix}\times 2-e^{-i4\theta}\\ \times 2-e^{-i4\theta}\end{matrix}$$

$$= \frac{2e^{i\theta}(2-e^{-i4\theta})}{4-2e^{-i4\theta}-2e^{i4\theta}+e^{i4\theta}(e^{-i4\theta})}$$

expand numerator and cancel on denominator

$$= \frac{4e^{i\theta}-2e^{-i3\theta}}{4+1-2(e^{i4\theta}+e^{-i4\theta})}$$

↳ notice we can use $e^{in\theta}+e^{-in\theta} = 2\cos n\theta$ on denominator

$$= \frac{4e^{i\theta}-2e^{-3i\theta}}{5-2(2\cos 4\theta)}$$

$$= \frac{4e^{i\theta}-2e^{-3i\theta}}{5-4\cos 4\theta}$$

now converting exponentials into mod-arg form

$$\frac{4(\cos\theta + i\sin\theta) - 2(\cos(-3\theta) + i\sin(-3\theta))}{5 - 4\cos 4\theta}$$

rewriting highlighted using cos even function - $\cos(-\theta) = \cos\theta$

and sin odd function - $\sin(-\theta) = -\sin\theta$

$$\Rightarrow \frac{4(\cos\theta + i\sin\theta) - 2\cos 3\theta + 2i\sin 3\theta}{5 - 4\cos 4\theta}$$

but question only asks for S i.e multiple of 'i':

$$\boxed{S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}}$$

WAY 2: working with mod-arg

$$C + iS = \frac{e^{i\theta}}{1 - \frac{1}{2}e^{i4\theta}}$$ ...into mod-arg: $$= \frac{\cos\theta + i\sin\theta}{1 - \frac{1}{2}(\cos\theta + i\sin\theta)^4} \times 2$$

$$= \frac{2(\cos\theta + i\sin\theta)}{2 - (\cos 4\theta + i\sin\theta)^4}$$

using DMT to get multi-angle trig function

$$= \frac{2(\cos\theta + i\sin\theta)}{2 - (\cos 4\theta + i\sin 4\theta)}$$

expand out

$$\frac{2\cos\theta + 2i\sin\theta}{2 - \cos 4\theta - i\sin 4\theta} = \frac{2\cos\theta + 2i\sin\theta}{(2 - \cos 4\theta) - i\sin\theta}$$

multiply by denominator but with 'i' multiple negated

$$= \frac{2\cos\theta + 2i\sin\theta}{(2 - \cos 4\theta) - i\sin\theta} \times \frac{(2 - \cos 4\theta) + i\sin 4\theta}{(2 - \cos 4\theta) + i\sin 4\theta}$$

expand numerator

$$\frac{4\cos\theta - 2\cos\theta\cos 4\theta + 2i\sin 4\theta + 4i\sin\theta - 2i\sin\theta\cos 4\theta - 2\sin\theta\sin 4\theta}{(4 - 4\cos 4\theta + \cos^2 4\theta) + \sin^2 4\theta}$$

$$= \frac{4\cos\theta + 4i\sin\theta - 2\cos\theta\cos 4\theta - 2\sin\theta\sin 4\theta + 2i\sin 4\theta\cos\theta - 2i\sin\theta\cos 4\theta}{5 - 4\cos 4\theta}$$

now compare imaginary terms

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

**Question 4 continued**

$$S = \frac{4\sin\theta + 2\sin4\theta\cos\theta - 2\sin\theta\cos4\theta}{5 - 4\cos4\theta}$$

using sine addition rule-numerator

$$= \frac{4\sin\theta + 2(\sin(4\theta - \theta))}{5 - 4\cos\theta} = \boxed{\frac{4\sin\theta + 2\sin3\theta}{5 - 4\cos\theta}}$$



**(Total for Question 4 is 8 marks)**

**5.** An engineer is investigating the motion of a sprung diving board at a swimming pool.
Let $E$ be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.
A diver jumps from the diving board.
The vertical displacement, $h$ cm, of the end of the diving board above $E$ is modelled by the differential equation

$$4\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$$

where $t$ seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation.

**(2)**

When $t = 0$, the end of the diving board is 20 cm below $E$ and is moving upwards with a speed of 55 cm s$^{-1}$.

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above $E$.

**(8)**

(c) Comment on the suitability of the model for large values of $t$.

**(2)**

(a) notice we are dealing with a homogenous 2ODE

A.E: $4m^2 + 4m + 37 = 0$

Solve for 'm': calc equtn solver or quadratic formula

$$m = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(37)}}{2(4)}$$

$$= \frac{-4 \pm \sqrt{-576}}{8}$$

$$= \frac{-4 \pm i\sqrt{576}}{8} = \frac{-\cancel{4}^{1} \pm \cancel{24}^{6} i}{\cancel{8}_{2}}$$

$$= -\frac{1}{2} \pm \frac{6}{2}i$$

$$\Rightarrow \quad m = -\frac{1}{2} \pm 3i$$

which suggests that there are 2 distinct complex roots $\therefore$ need to use general formula $e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

$\boxed{\text{G.S:}}$ $\boxed{h = e^{-0.5t}(A\cos 3t + B\sin 3t)}$

Question 5 continued

(b) subbing the initial conditions into G.S (to get a particular solution)

at $t=0, h=-20$

$$-20 = e^{0}(A\cos(0) + B\sin(0))$$

$$\Rightarrow \boxed{A = -20}$$

next differentiate so use speed - using $\frac{d}{dt}(e^{kt}) = ke^{kt}$

product rule

$$\frac{dh}{dt} = -0.5e^{-0.5t}(A\cos 3t + B\sin 3t) + e^{0.5t}(-3A\sin 3t + 3B\cos 3t)$$

at $t=0, \frac{dh}{dt} = 55$

$$55 = -0.5e^{-0.5(0)}(A\cos(3\times 0) + B\sin(3\times 0))$$

$$+ e^{0.5(0)}(-3A\sin(3\times 0) + 3B\cos(3\times 0))$$

$$\Rightarrow 55 = -0.5A + 3B$$

subbing in $A = -20$

$$55 = -0.5(-20) + 3B$$

$$\Rightarrow 3B = 45$$

$\div 3$ $\div 3$

$$\boxed{B = 15}$$

$\therefore$ P.S $h = e^{-0.5t}(-20\cos 3t + 15\sin 3t)$

but asked for max displacement i.e $\frac{dh}{dt} = 0$ - this will help us to find the 't' at which $h_{max}$ occurs

differentiate P.S using PRODUCT RULE and $\frac{d}{dt}(\sin kt) = k\cos kt$

$\frac{d}{dt}(\cos kt) = -\sin kt$

$$= -0.5e^{0.5t}(-20\cos 3t + 15\sin 3t) + e^{-0.5t}(60\sin 3t + 45\cos 3t)$$

$$= 0$$

multiply brackets by -0.5

$$\Rightarrow e^{-0.5t}(10\cos 3t - 7.5\sin 3t) + e^{-0.5t}(60\sin 3t + 45\cos 3t) = 0$$

collect like trig

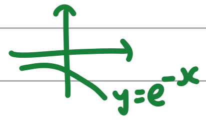
$$e^{-0.5t}(55\cos 3t + 52.5\sin 3t) = 0$$

Question 5 continued

making each bracket equal 0

$e^{-0.5t} \neq 0$

due to exponential graph properties




$55\cos 3t + 52.5\sin 3t = 0$

$\div \cos 3t \qquad \div \cos 3t$

$\Rightarrow 55 + 52.5\tan 3t = 0$

$\Rightarrow 52.5\tan 3t = -55$

$\div 52.5 \qquad \div 52.5$

$\Rightarrow \tan 3t = -\frac{55}{52.5}$

arctan of both sides

$\Rightarrow 3t = \tan^{-1}\left(-\frac{55}{52.5}\right)$

but time > 0 $\therefore$ using tan angle law

$3t = \tan^{-1}\left(-\frac{55}{52.5}\right) + \pi$

$\Rightarrow 3t = 2.33294..$

$\div 3 \qquad \div 3$

$\boxed{t = 0.7776...}$

now subbing this 't' into our P.S to get the value of $h_{max}$

$\Rightarrow h_{max} = e^{0.5(0.776...)}\left(-20\cos(3\times 0.77...) + 15\sin(3\times 0.77..)\right)$

$= 16.715...$

$\boxed{= 16.7\text{cm } (3\text{ s.f})}$

(c) value of h is very small when t is large (-ve exponential) which considering the context of the question is likely to be correct ; displacement should decrease AND never = 0 $\boxed{\therefore \text{ suitable model}}$



P 6 1 1 7 8 A 0 1 4 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 5 continued

(Total for Question 5 is 12 marks)

**6.** In an Argand diagram, the points $A$, $B$ and $C$ are the vertices of an equilateral triangle with its centre at the origin. The point $A$ represents the complex number $6 + 2\mathrm{i}$.

(a) Find the complex numbers represented by the points $B$ and $C$, giving your answers in the form $x + \mathrm{i}y$, where $x$ and $y$ are real and exact.

**(6)**

The points $D$, $E$ and $F$ are the midpoints of the sides of triangle $ABC$.

(b) Find the exact area of triangle $DEF$.

**(3)**

(a) using knowledge from Yr 2 complex numbers - if $z_1$ is one root of the equation $z^n = s$ and $1, w, w^2, \ldots, w^{n-1}$ are the nth roots of unity, then the roots of $z^n = s$ are given by $z_1, z_1 w, z_1 w^2, \ldots, z_1 w^{n-1}$

$\Rightarrow$ because we're looking at an equilateral triangle, we can find the coordinates $B$ and $C$ by multiplying $z_1 = 6+2i$ by the cube roots of unity

following general formula: $w = e^{i2\pi/k}$

$\Rightarrow w = e^{i2\pi/3}$

METHOD 1: working with exponentials and linear transformations

changing $6+2i$ into exponential form

$|6+2i| = \sqrt{(6)^2+(2)^2}$

$= \sqrt{36+4}$

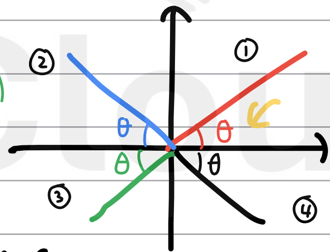
$= \sqrt{40}$

$\arg(6+2i) = \tan^{-1}(2/6)$

$= \boxed{\tan^{-1}(1/3)}$

↳ keep as this as then easier to use in calculations

① $\theta$
② $\pi-\theta$
③ $-(\pi-\theta)$
④ $-\theta$




$z_1 = \sqrt{40}\, e^{i\tan^{-1}(1/3)}$

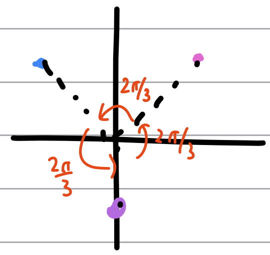
$\Rightarrow B = \sqrt{40}\, e^{i\tan^{-1}(1/3)} \times e^{2\pi/3\, i}$

$= \sqrt{40}\, e^{i(\tan^{-1}(1/3) + 2\pi/3)}$

but notice that if were to convert this $B$ into $a+bi$ form we won't get exact values

$\therefore$ noticing that we should interpret the multiplying by $e^{2\pi/3\, i}$ as rotating $z_1$ by $\frac{2(180°)}{3} = \boxed{120°}$ (indicated below)

**Question 6 continued**




hence using our knowledge from Year 1 of linear transformations

rotation by $\theta°$ anticlockwise : $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

subbing in $\theta = 120°$ and multiply by $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} \cos(120°) & -\sin(120°) \\ \sin(120°) & \cos(120°) \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

matrix multiplication - "rows into columns"

$$\begin{pmatrix} 6(\cos(120°)) - 2(\sin(120°)) \\ 6(\sin(120°)) + 2(\cos(120°)) \end{pmatrix}$$

evaluate on calc CLASSWIZ

$$\begin{pmatrix} -3-\sqrt{3} \\ -1+3\sqrt{3} \end{pmatrix}$$

because above is in the format $\begin{pmatrix} x \\ y \end{pmatrix}$, we can say that $B$ is in the form $x+iy$

$$\Rightarrow \boxed{B = (-3-\sqrt{3}) + i(-1+3\sqrt{3})}$$

...now for $C$, 2 ways:

WAY 1: using $z_1$

$$C = z_3 = \sqrt{40}e^{i\tan^{-1}(1/3)} \times \left(e^{i2\pi/3}\right)^2$$

$$= \sqrt{40}e^{i(\tan^{-1}(1/3) + \frac{4\pi}{3})}$$

which is the same as rotating $6+2i$ by $\frac{4(180°)}{3} = 240°$ about origin

$$\begin{pmatrix} \cos(240°) & -\sin(240°) \\ \sin(240°) & \cos(240°) \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

matrix multiplication - "rows into columns"

$$\begin{pmatrix} 6\cos(240°) - 2\sin(240°) \\ 6\sin(240°) + 2\cos(240°) \end{pmatrix}$$

eval on calc

$$= \begin{pmatrix} -3+\sqrt{3} \\ -1-3\sqrt{3} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$ ...in $x+iy$ form:

$$\boxed{\therefore C = (-3+\sqrt{3}) + i(-1-3\sqrt{3})}$$

WAY 2: using $z_2$

rotating $z_2$ by another $120°$ - represented below:

$$\begin{pmatrix} \cos(120°) & -\sin(120°) \\ \sin(120°) & \cos(120°) \end{pmatrix}\begin{pmatrix} -3-\sqrt{3} \\ -1+3\sqrt{3} \end{pmatrix}$$

matrix multiplication - "rows into columns"

$$\begin{pmatrix} (-3-\sqrt{3})\cos(120°) + (-1+3\sqrt{3})(-\sin(120°)) \\ (-3-\sqrt{3})\sin(120°) + (-1+3\sqrt{3})\cos(120°) \end{pmatrix}$$

evaluate on calc

$$\begin{pmatrix} \frac{3+\sqrt{3}}{2} + \frac{-9+\sqrt{3}}{2} \\ -\left(\frac{3+3\sqrt{3}}{2}\right) + \left(\frac{1-3\sqrt{3}}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{-6+2\sqrt{3}}{2} \\ \frac{-2-6\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -3+\sqrt{3} \\ -1-3\sqrt{3} \end{pmatrix}$$

$$\Rightarrow \boxed{C = (-3+\sqrt{3}) + i(-1-3\sqrt{3})}$$

METHOD 2: working with a+bi form

instead of using exponential form of $w = e^{i2\pi/3}$ - can convert this into mod-arg form $\rightarrow$ a+bi form and then multiplying becomes a simple case of expanding brackets

$$w = e^{i2\pi/3}$$
$$= \cos(2\pi/3) + i\sin(2\pi/3)$$
$$= -1/2 + i\left(\frac{\sqrt{3}}{2}\right)$$

now multiplying

$$z_1 = 6+2i$$
$$B = z_1 w = (6+2i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

expand brackets

$$= -3 + \left(\frac{6\sqrt{3}}{2} - 1\right)i - \sqrt{3}$$

$$\Rightarrow \boxed{B = (-3-\sqrt{3}) + i(3\sqrt{3}-1)}$$

... for $C$ - two ways:

WAY 1: using $z_1$

$$C = z_1 w^2 = (6+2i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2$$

on calc

$$\Rightarrow \boxed{C = (-3+\sqrt{3}) + i(-1+3\sqrt{3})}$$

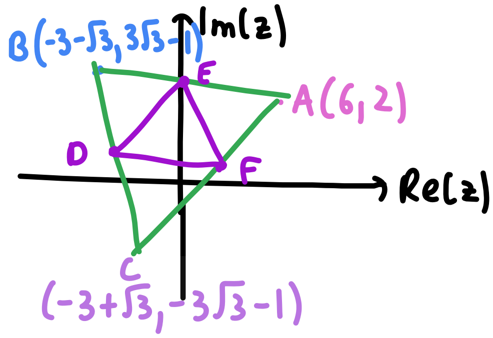
WAY 2: using $z_2$

$$C = (-3-\sqrt{3}) + i(-1+3\sqrt{3})\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

on calc

$$\Rightarrow \boxed{C = \left(-3+\sqrt{3}+i(-1-3\sqrt{3})\right)}$$

(b) illustrating ABC on a sketch - together with m.ps



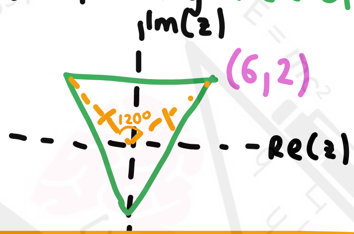

need area(DEF)

WAY 1: (best if spot):

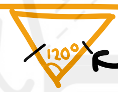
$$\boxed{\Delta DEF = \frac{1}{4}\triangledown}$$

∴ using coordinates given in prev. parts

hence finding area of ABC




$$\boxed{= 3 \times \triangledown}$$

now finding  $\leftarrow \sqrt{6^2+2^2}$

$= \sqrt{36+4}$

$= \sqrt{40}$

∴ using sine rule

$\frac{1}{2}(\sqrt{40})(\sqrt{40})\sin(120°)$

$= 20\sin(120°)$

$= 10\sqrt{3}$

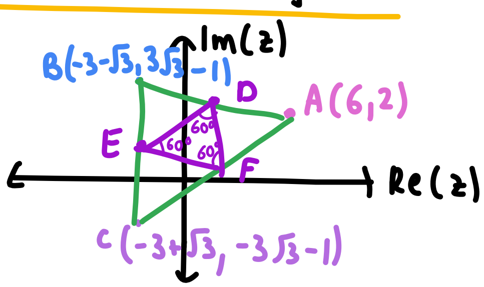
$\therefore \triangledown = 3 \times 10\sqrt{3}$

$= 30\sqrt{3}$

$\Rightarrow \Delta DEF = \frac{1}{4}(30\sqrt{3})$

$= \frac{15}{2}\sqrt{3}$ units$^2$

METHOD 2: using m.ps




finding each of the midpoints: use formula

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

... for D:

$$\left(\frac{-3-\sqrt{3}+6}{2}, \frac{3\sqrt{3}-1+2}{2}\right)$$

... for E:

$$\left(\frac{-3-\sqrt{3}+(-3+\sqrt{3})}{2}, \frac{3\sqrt{3}-1+(-3\sqrt{3}-1)}{2}\right)$$

**Question 6 continued**

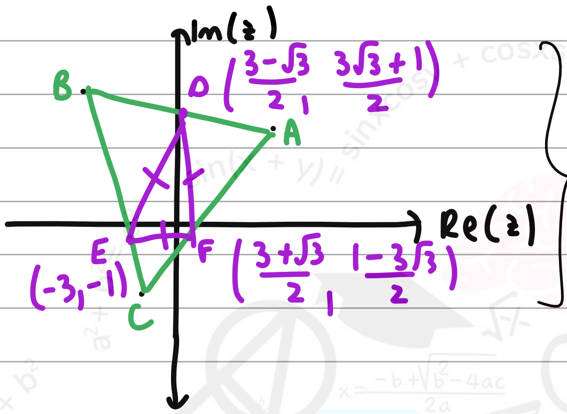
$$= \left(\frac{3-\sqrt{3}}{2}, \frac{3\sqrt{3}+1}{2}\right) \qquad = \left(-\frac{6}{2}, \frac{-2}{2}\right) = (-3,-1)$$

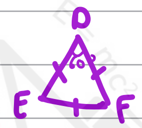
...for F:

$$\left(\frac{6+(-3+\sqrt{3})}{2}, \frac{2+(-3\sqrt{3}-1)}{2}\right)$$

$$= \left(\frac{3+\sqrt{3}}{2}, \frac{1-3\sqrt{3}}{2}\right)$$

marking these on sketch:




- can use sine rule - but need magnitude of any of the three sides DE, DF, EF - all equal to each other

eg. FD:

$$\sqrt{\left(\frac{3+\sqrt{3}}{2} - \left(\frac{3-\sqrt{3}}{2}\right)\right)^2 + \left(\frac{1-3\sqrt{3}}{2} - \frac{3\sqrt{3}+1}{2}\right)^2}$$

$$= \sqrt{(\sqrt{3})^2+(-3\sqrt{3})^2} = \sqrt{3+27} = \sqrt{30} = DE$$

$\therefore$ area $\Delta DEF$:

$$\frac{1}{2} \times \sqrt{30} \times \sqrt{30} \times \sin(60°)$$

$$\boxed{= \frac{15\sqrt{3}}{2} \text{ units}^2}$$

**(Total for Question 6 is 9 marks)**



P61178A01928

**7.**

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Find the values of $k$ for which the matrix **M** has an inverse.

**(2)**

(b) Find, in terms of $p$, the coordinates of the point where the following planes intersect

$$2x - y + z = p$$
$$3x - 6y + 4z = 1$$
$$3x + 2y - z = 0$$

**(5)**

(c) (i) Find the value of $q$ for which the set of simultaneous equations

$$2x - y + z = 1$$
$$3x - 5y + 4z = q$$
$$3x + 2y - z = 0$$

can be solved.

(ii) For this value of $q$, interpret the solution of the set of simultaneous equations geometrically.

**(4)**

(a) remembering that for matrix M to have an inverse means it has to be non-singular i.e det(M) $\neq 0$ - hence let's find the value of 'k' for which M is singular so that $k \neq$ this value

↳ finding det(M) - 3×3 matrix

$$\det(M) = 2\begin{vmatrix} k & 4 \\ 2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & k \\ 3 & 2 \end{vmatrix}$$

$= 2(-k-8) + 1(-3-12) + 1(6-3k)$

expand above

$= -2k - 16 - 3 - 12 + 6 - 3k$

collect like terms

$\Rightarrow -5k - 25 = 0$

$\Rightarrow 5k = -25$

$\div 5 \qquad \div 5$

$\Rightarrow k = -5$

$\therefore$ for M to have an inverse, $\boxed{k \neq -5}$

(b) WAY 1: using matrices

**Question 7 continued**

notice given a system of linear equtns - hence to find p.o.i, need to solve for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

↳ using general formula for matrix equtns : $Mx = y$ - splitting into matrix of coefficients, of variables and of constants

$$\Rightarrow \begin{pmatrix} 2 & -1 & 1 \\ 3 & -6 & 4 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$$

now solve for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ ∴ multiply LHS of each side of the equation by $M^{-1}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -6 & 4 \\ 3 & 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$$

↳ notice in exam situation can evaluate the inverse on calc CLASSWIZ (no unknowns)

↳ Matrix - Dfn matrix - 1. Mat A - 3x3 - OPT - Matrix calc - Mat A - $\boxed{x^{-1}}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2/5 & 1/5 & 2/5 \\ 3 & -1 & -1 \\ \frac{24}{5} & -7/5 & -9/5 \end{pmatrix} \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$$

factorise 1/5 out

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 & 1 & 2 \\ 15 & -5 & -5 \\ 24 & -7 & -9 \end{pmatrix} \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix}$$

RHS - matrix multiplication - "rows into columns"

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2(p) + 1(1) + 2(0) \\ 15(p) - 5(1) - 5(0) \\ 24(p) - 7(1) - 9(0) \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -2p+1 \\ 15p-5 \\ 24p-7 \end{pmatrix}$$

$$\Rightarrow \left( \frac{-2p+1}{5}, \; 3p-1, \; \frac{24p-7}{5} \right)$$

WAY 2: algebraically - solving 3 variable sim. equtns for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Question 7 continued

$$2x - y + z = p \quad -①$$
$$3x - 6y + 4z = 1 \quad -②$$
$$3x + 2y - z = 0 \quad -③$$

eg. eliminate 'x' from ① and ②

①×3 −②×2
$$\begin{array}{r} 6x - 3y + 3z = 3p \\ - \quad 6x - 12y + 8z = 2 \\ \hline 9y - 5z = 3p - 2 \quad -④ \end{array}$$

and eliminate 'x' from ② and ③

②−③
$$\begin{array}{r} 3x - 6y + 4z = 1 \\ - \quad 3x + 2y - z = 0 \\ \hline -8y + 5z = 1 \end{array}$$
or $8y - 5z = -1 \quad -⑤$

now eliminate 'z' from ④ and ⑤

④−⑤
$$\begin{array}{r} 9y - 5z = 3p - 2 \\ - \quad 8y - 5z = -1 \\ \hline y = 3p - 1 \end{array}$$

subbing y into ⑤ (the one without unknown 'p') to get z

$$8(3p - 1) - 5z = -1$$

expand

$$24p - 8 - 5z = -1$$
$$\Rightarrow 5z = 24p - 8 + 1$$
$$5z = 24p - 7$$
$$\div 5 \qquad \div 5$$
$$\Rightarrow z = \frac{24p - 7}{5}$$

now subbing y and z into ANY of ①,② and ③ for 'x'

$$3x + 2(3p - 1) - \left(\frac{24p - 7}{5}\right) = 0$$
$$\Rightarrow 3x + 6p - 2 - \frac{24p + 7}{5} = 0$$
$$\Rightarrow 3x = -6/5\, p + 3/5$$
$$\div 3 \qquad \div 3$$
$$\Rightarrow x = -2/5\, p + 1/5$$

$$\Rightarrow \text{p.o.i} = \boxed{\left(\frac{-2p+1}{5},\ 3p-1,\ \frac{24p-7}{5}\right)}$$

(c)(i) 'can be solved' suggests that the system of linear equations is consistent i.e that there is at least one set of values that satisfies all equations simultaneously

METHOD 1: eliminate 'z'

↳ to prove consistency need to eliminate eg. 'z' out of any 2 pairs of linear equtns

$$2x - y + z = 1 \quad -①$$
$$3x - 5y + 4z = 2 \quad -②$$
$$3x + 2y - z = 0 \quad -③$$

eliminate 'z' from ① and ②

$4\times① - ②$

$$\begin{array}{r} 8x - 4y + 4z = 4 \\ -\ \ 3x - 5y + 4z = q \\ \hline 5x + y = 4 - q \quad -④ \end{array}$$

WAY 1: and similarly from ① and ③

① + ③

$$\begin{array}{r} 2x - y + z = 1 \\ +\ \ 3x + 2y - z = 0 \\ \hline 5x + y = 1 \quad -⑤ \end{array}$$

but the ④ and ⑤ have to be consistent so given LHS of both are equal, RHS also have to be equal

$\Rightarrow 4 - q = 1$

$\Rightarrow \boxed{q = 3}$

WAY 2: similarly for ② and ③

② + 4×③

$$\begin{array}{r} 3x - 5y + 4z = q \\ +\ \ 12x + 8y - 4z = 0 \\ \hline 15x + 3y = q \end{array}$$

but ④ and ⑤ have to be consistent - if LHS share a scalar multiple of 3, then ④'s RHS ×3 should equal RHS of ⑤

$3(4 - q) = q$

expand

$12 - 3q = q$

$\Rightarrow 4q = 12$

$\div 4 \quad \div 4$

$\boxed{q = 3}$

METHOD 2: allocating a number to a variable

$$2x - y + z = 1 \quad -①$$
$$3x - 5y + 4z = q \quad -②$$
$$3x + 2y - z = 0 \quad -③$$

... to solve, let $x = 1$:

↳ solving ① and ③

$$2(1) - y + z = 1 \quad -①$$
$$3(1) + 2y - z = 0 \quad -③$$

... solve by elimination :

① + ③

$$\begin{array}{r} -y + z = -1 \\ +\ \ 2y - z = -3 \\ \hline \end{array}$$

Question 7 continued

$\underline{\overline{y=-4}}$

sub into any of ① or ②

$-(-4)+z=-1$

$4+z=-1$

$\Rightarrow z=-5$

because all three equations share a common $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then if sub above into ② - should give desired value of 'q'

$3(1)-5(-4)+4(-5)=q$

$\Rightarrow 3+20-20=q$

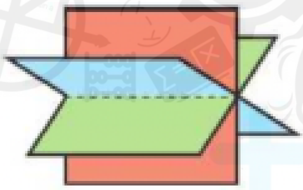
$\Rightarrow \boxed{q=3}$

(ii) if consistent, means only 2 geometrical interpretations:

- same plane
- sheaf

but considering ①, ② and ③ straight away we can see that these are NOT scalar multiples of each other ∴ NOT same plane

$\Rightarrow$ $\boxed{\text{form a sheaf}}$



(Total for Question 7 is 11 marks)

**8.**

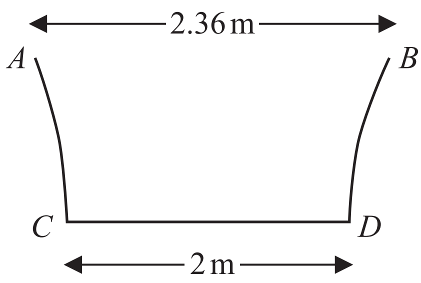



**Figure 1**

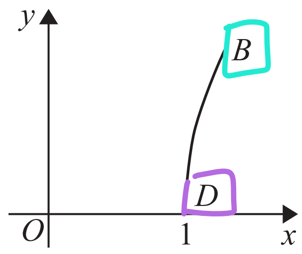



**Figure 2**

Figure 1 shows the central vertical cross section $ABCD$ of a paddling pool that has a circular horizontal cross section. Measurements of the diameters of the top and bottom of the paddling pool have been taken in order to estimate the volume of water that the paddling pool can contain.

Using these measurements, the curve $BD$ is modelled by the equation

$$y = \ln(3.6x - k) \qquad \boxed{1 \leqslant x \leqslant 1.18}$$

as shown in Figure 2.

(a) Find the value of $k$. **(1)**

(b) Find the depth of the paddling pool according to this model. **(2)**

The pool is being filled with water from a tap.

(c) Find, in terms of $h$, the volume of water in the pool when the pool is filled to a depth of $h$ m. → LIMITS **(5)**

Given that the pool is being filled at a constant rate of 15 litres every minute,

(d) find, in cm h$^{-1}$, the rate at which the water level is rising in the pool when the depth of the water is 0.2 m. **(3)**

(a) looking at Fig 2 – see how the coordinates of D would be (1,0)

↳ subbing these into the equation for curve BD

$0 = \ln(3.6(1) - k)$

$0 = \ln(3.6 - k)$

from log properties know that logs are equal to 0 only for $\ln(1)$ ∴ $3.6 - k = 1$

$\Rightarrow k = 3.6 - 1$

$\Rightarrow \boxed{k = 2.6}$

$\therefore y = \ln(3.6x - 2.6)$

DO NOT WRITE IN THIS AREA

**Question 8 continued**

(b) finding the depth requires us to find the 'y' coordinate of B

$\therefore$ Sub in $x = 1.18$ (from range) into curve BD

$$\Rightarrow y = \ln(3.6(1.18) - 2.6)$$

$$= \ln(1.648)$$

$$\Rightarrow y = 0.49956\ldots$$

$$\Rightarrow \boxed{\text{depth} = 0.500\text{m}} \text{ (to 3 s.f)}$$

(c) to find the volume of water, need to use formula for volume of revolution around the 'y-axis':

$$V = \pi \int_{\alpha}^{\beta} x^2 \, dy$$

but our BD is given as an expression in 'y' $\therefore$ rearrange for 'x':

$$e^y = 3.6x - 2.6$$

$$\Rightarrow 3.6x = e^y + 2.6$$

$\div 3.6$ $\div 3.6$

$$x = \frac{e^y + 2.6}{3.6}$$

$$\Rightarrow x^2 = \left(\frac{e^y + 2.6}{3.6}\right)^2$$

subbing into integral with coefficient out in front:

$$\frac{\pi}{3.6^2} \int_0^h (e^y + 2.6)^2 \, dy$$

expand brackets

$$\frac{\pi}{3.6^2} \int_0^h (e^{2y} + 5.2e^y + 6.76) \, dy$$

integrate - use $\int e^{ky} dy = \frac{1}{k} e^{ky} + c$

$$\frac{\pi}{3.6^2} \left[0.5e^{2y} + 5.2e^y + 6.76y\right]_0^h$$

$$= \frac{\pi}{3.6^2} \left\{ \left[0.5e^{2h} + 5.2e^h + 6.76h\right] - \left[0.5 + 5.2 + 6.76(0)\right] \right\}$$

$$= \frac{\pi}{3.6^2} \left\{ \left[0.5e^{2h} + 5.2e^h + 6.76h - 5.7\right] \right\}$$

$\div 3.6^2$

Question 8 continued $\Rightarrow \boxed{V = \pi\left(\frac{25}{648}e^{2h} + \frac{65}{162}e^{h} + \frac{169}{324}h - \frac{95}{216}\right) m^3}$

(d) interpret 'rate at which water level is rising' as $\frac{dh}{dt}$ and 'pool is being filled at a rate of 15L/m' as $\frac{dV}{dt} = 15\frac{\ell}{min}$

but need in $cmh^{-1}$ so

$$\frac{15}{1000} \times 60 = 0.9$$

∴ using connected rates of change

need $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

given $\frac{dV}{dt}$ ∴ need to find $\frac{dh}{dV}$

WAY 1: differentiate part (c) and $\frac{1}{Ans}$ to get $\frac{dh}{dV}$

$$V = \frac{\pi}{3.6^2}\left(0.5e^{2h} + 5.2e^{h} + 6.76h - 5.7\right)$$

$$\frac{dV}{dh} = \frac{\pi}{3.6^2}\left(e^{2h} + 5.2e^{h} + 6.76\right)$$

when $h = 2$,

$$= \frac{\pi}{3.6^2}\left(e^{2(0.2)} + 5.2e^{0.2} + 6.76\right)$$

$$= 3.539\ldots \, cmh^{-1}$$

$$\therefore \frac{dh}{dV} = \frac{1}{3.539\ldots}$$

sub into connected rate of change:

$$\frac{dh}{dt} = \frac{1}{3.539\ldots} \times \frac{15}{1000} \times 60$$

$$= 0.254\ldots$$

$$\boxed{= 0.254 \, cmh^{-1} \, (3 \, s.f)}$$

WAY 2: interpreting 'depth' as 'y' for which $x = 0.2$ - sub into rearranged equation for curve BD

$$x = \frac{e^{0.2} + 2.6}{3.6}$$

$$\Rightarrow A = \pi\left(\frac{e^{0.2} + 2.6}{3.6}\right)^2 = 3.54$$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

**Question 8 continued**

$\therefore V = \frac{1}{3.54..} \times 0.9$

$= 0.2542... \ mh^{-1}$

$\boxed{= 25.4 \ cmh^{-1}}$

**Question 8 continued**

**(Total for Question 8 is 11 marks)**

**TOTAL FOR PAPER IS 75 MARKS**